INFORMATION AND CONTROL 26, 116–133 (1974)

Sets Constructed by Acceptors

MIROSLAV NOVOTNÝ

Mathematical Institute of the Czechoslovak Academy of Sciences, Branch Brno, Czechoslovakia

Sets of both finite and infinite sequences can be constructed by acceptors; special cases of these acceptors and the structure of the corresponding constructed sets were studied by Mezník (1972). We give a complete characterization of sets constructed by acceptors of three special classes and a new complete characterization of regular sets.

INTRODUCTION

Pawlak (1969) defined deterministic machines which were able to construct sets of both finite and infinite sequences. Kwasowiec (1970) investigated the structure of sets constructed by Pawlak's machines. Mezník (1972) gave a nondeterministic variant of such machines and studied the structure of sets constructed by them. Another nondeterministic variant of these machines was studied by Kwasowiec (1970a).

A machine of Mezník is a finite acceptor of a special class and the set constructed by such an acceptor consists of all finite sequences which are accepted in the usual way and of all infinite sequences which are accepted in a generalized way by the acceptor. We give some results on general acceptors and introduce special classes of acceptors. The structure of sets constructed by acceptors of each of these classes is described. Each of these constructed sets can be expressed in the form of a sum of two summands where the first is the set accepted by the given acceptor and the second is the set of limits corresponding to sequences of elements which are taken from the set accepted by a suitable acceptor. The limit of an infinite sequence of finite sequences is defined in a very natural way. We are able to describe these two summands without using the terminology of acceptors and to characterize the sets constructed by acceptors of our classes completely in this way. Simultaneously, we obtain a complete characterization of regular sets: They are obtained from sets accepted by finite acceptors of one of these classes by means of length-preserving homomorphisms.

1. Acceptors

1.1. *Notation*

We denote by Ord the class of all ordinals, by ω the least infinite ordinal. Finite ordinals are called natural numbers; we denote by N the set of all natural numbers.

Let V be a set, n an ordinal, $n \leqslant \omega$. Then each mapping a of the set $\{i;\ i \in \text{Ord},\ i < n\}$ into V is called a sequence of the type n formed of elements of the set V. If $n = 0$ then a is the empty mapping which will be denoted by Λ. If $n > 0$ then we write a_i instead of $a(i)$ and the sequence is denoted by $(a_i)_{0 \leqslant i < n}$. It is advantageous to put $(a_i)_{0 \leqslant i < 0} = \Lambda$. We also write $(a_0, a_1, \ldots, a_{n-1})$ instead of $(a_i)_{0 \leqslant i < n}$ if $0 < n < \omega$. Sequences of finite types are called strings; the type of a string is also its length; if x is a string then we denote its length by $|x|$.

If V is a set and n an ordinal, $0 \leqslant n \leqslant \omega$, then we denote by V^n the set of all sequences of the type n which are formed of elements of the set V. We put $V^* = \bigcup_{0 \leqslant n < \omega} V^n$, $V^\infty = \bigcup_{0 \leqslant n \leqslant \omega} V^n$. We identify each sequence $(a_i)_{0 \leqslant i < 1}$ with the element a_0; thus, we have $V \subseteq V^* \subseteq V^\infty$. In V^*, we define the binary operation of concatenation which assigns, to any sequences $a = (a_i)_{0 \leqslant i < m}$, $b = (b_i)_{0 \leqslant i < n}$, the sequence $ab = (c_i)_{0 \leqslant i < m+n}$ where $c_i = a_i$ for $0 \leqslant i < m$ and $c_i = b_{i-m}$ for $m \leqslant i < m + n$. Then V^* is a monoid, i.e., a set with an associative binary operation (concatenation) and with a nullary operation (Λ); V^* is called to be the free monoid on V. From the associativity of the concatenation and from the identification of $a \in V$ with $(a) \in V^*$, we have $(a_i)_{0 \leqslant i < n} = (a_0, a_1, \ldots, a_{n-1}) = (a_0)(a_1) \cdots (a_{n-1}) = a_0 a_1 \cdots a_{n-1}$ for each $0 < n < \omega$ and any $a_0, a_1, \ldots, a_{n-1} \in V$.

Let V be a set, k, n ordinals such that $0 \leqslant k \leqslant n \leqslant \omega$, $k < \omega$; suppose $b_i \in V$ for each i such that $k \leqslant i < n$. Then, by $(b_i)_{k \leqslant i < n}$, we mean the sequence $(a_i)_{0 \leqslant i < n-k}$ where $a_i = b_{i+k}$ for each i, $0 \leqslant i < n - k$, if $n < \omega$; if $n = \omega$ then we define $(b_i)_{k \leqslant i < n}$ to be the sequence $(a_i)_{0 \leqslant i < \omega}$ where $a_i = b_{i+k}$ for each i, $0 \leqslant i < \omega$.

If V, U are sets and $\varphi\colon V \to U$ a mapping then we denote by φ_* the mapping of V^* into U^* which is defined as follows. $\varphi_*(\Lambda) = \Lambda$, $\varphi_*(a_0 a_1 \cdots a_m) = \varphi(a_0)\,\varphi(a_1) \cdots \varphi(a_m)$ where $m \geqslant 0$ is a natural number and $a_i \in V$ for each i, $0 \leqslant i \leqslant m$. Clearly, φ_* is a homomorphism of V^* into U^* such that $|\varphi_*(x)| = |x|$ for each $x \in V^*$. A homomorphism Φ of V^* into U^* such that $|\Phi(x)| = |x|$ for each $x \in V^*$ is said to be length-preserving. It is easy to see that, to each length-preserving homomorphism Φ of V^* into U^*, there exists a mapping φ of V into U such that $\Phi = \varphi_*$.

We denote by 2^V the set of all subsets of the set V.

1.2. Definition. Let V be a set, $x = (x_i)_{0 \leqslant i < \omega} \in V^\omega$ a sequence, $x^j \in V^*$ a sequence for each $j \in N$. We say that x is a *limit* of the sequence $(x^j)_{0 \leqslant j < \omega}$ if the following condition is satisfied. There is an increasing sequence of natural numbers $(n^j)_{0 \leqslant j < \omega}$ such that $x^j = (x_i)_{0 \leqslant i < n^j}$. If x is a limit of the sequence $(x^j)_{0 \leqslant j < \omega}$ then we write $x = \lim_{j \to \omega} x^j$.

1.3. Definition. Let V be a set, suppose $M \subseteq V^*$, $x \in V^\omega$. Then we put $x \in \mathscr{D}(M)$ iff there exists a sequence $(x^j)_{0 \leqslant j < \omega}$ of elements of the set M such that $x = \lim_{j \to \omega} x^j$.

1.4. Definition. An *acceptor* is a 5-tuple $\mathfrak{N} = \langle S, V, f, I, F \rangle$ where S, V are sets, f a mapping of $S \times V$ into 2^S and I, F subsets of S.

1.5. Definition. Let $\mathfrak{N} = \langle S, V, f, I, F \rangle$ be an acceptor, $x = (x_i)_{1 \leqslant i < n} \in V^*$ a sequence where $0 < n < \omega$. We put $x \in \mathscr{A}(\mathfrak{N})$ iff there is a sequence $(s_i)_{0 \leqslant i < n}$ of elements of S such that $s_0 \in I$, $s_i \in f(s_{i-1}, x_i)$ for each i satisfying $1 \leqslant i < n$ and $s_{n-1} \in F$. The sequences of the set $\mathscr{A}(\mathfrak{N})$ are said to be *accepted* by $\mathfrak{N}$; the set $\mathscr{A}(\mathfrak{N})$ is said to be accepted by $\mathfrak{N}$, too.

1.6. Definition. Let $\mathfrak{N} = \langle S, V, f, I, F \rangle$ be an acceptor, $x = (x_i)_{1 \leqslant i < \omega} \in V^\omega$ a sequence. We put $x \in \mathscr{B}(\mathfrak{N})$ iff there is a sequence $(s_i)_{0 \leqslant i < \omega}$ of elements of S such that $s_0 \in I$ and $s_i \in f(s_{i-1}, x_i)$ for each i satisfying $1 \leqslant i < \omega$. We put $\mathscr{C}(\mathfrak{N}) = \mathscr{A}(\mathfrak{N}) \cup \mathscr{B}(\mathfrak{N})$. The sequences of the set $\mathscr{C}(\mathfrak{N})$ are said to be *constructed* by $\mathfrak{N}$; the set $\mathscr{C}(\mathfrak{N})$ is said to be constructed by $\mathfrak{N}$, too.

1.7. Definition. Let $\mathfrak{N} = \langle S, V, f, I, F \rangle$ be an acceptor. Then we put $\mathfrak{t}(\mathfrak{N}) = \langle S, V, f, I, S \rangle$.

1.8. Lemma. *Let $\mathfrak{N}$ be an acceptor. Then $\mathscr{C}(\mathfrak{N}) \subseteq \mathscr{A}(\mathfrak{N}) \cup \mathscr{D}(\mathscr{A}(\mathfrak{t}(\mathfrak{N})))$.*

Proof. Suppose $x = (x_i)_{1 \leqslant i < n} \in \mathscr{C}(\mathfrak{N})$, $0 < n \leqslant \omega$. Then there exists a sequence $(s_i)_{0 \leqslant i < n}$ of elements of S such that $s_0 \in I$, $s_i \in f(s_{i-1}, x_i)$ for each i with the property $1 \leqslant i < n$ and $s_{n-1} \in F$ if $n < \omega$. If $n < \omega$ then we have, clearly, $x \in \mathscr{A}(\mathfrak{N})$. If $n = \omega$ then we put $x^j = (x_i)_{1 \leqslant i < j+1}$ for each $j \in N$. Thus, $x^j \in \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$ for each $j \in N$ and $\lim_{j \to \omega} x^j = x$. It follows $x \in \mathscr{D}(\mathscr{A}(\mathfrak{t}(\mathfrak{N})))$.

1.9. Example. We put $S(n) = \{(n, i);\ i \in N,\ 0 \leqslant i \leqslant n\}$ for each natural number $n \in N$, $S = \bigcup_{0 \leqslant n < \omega} S(n)$, $I = \{(n, 0);\ n \in N\}$, $F = \{(n, n);\ n \in N\}$.

We put, for $n, i, j \in N$, $0 \leqslant i \leqslant n$,

$$f((n, i), j) = \begin{cases} \varnothing & \text{if } i \neq j \quad \text{or } j = i = n, \\ \{(n, i+1)\} & \text{if } j = i < n. \end{cases}$$

We define $\mathfrak{N} = \langle S, N, f, I, F \rangle$.

Clearly, $(i)_{0 \leqslant i < n} \in \mathscr{A}(\mathfrak{N}) \subseteq \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$ for each $n \in N$. It follows $(i)_{0 \leqslant i < \omega} \in \mathscr{D}(\mathscr{A}(\mathfrak{t}(\mathfrak{N})))$. Clearly, $\mathscr{C}(\mathfrak{N})$ contains only sequences of finite types which implies $(i)_{0 \leqslant i < \omega} \notin \mathscr{C}(\mathfrak{N})$. We have proved $\mathscr{C}(\mathfrak{N}) \neq \mathscr{A}(\mathfrak{N}) \cup \mathscr{D}(\mathscr{A}(\mathfrak{t}(\mathfrak{N})))$.

In Sections 2 and 3, we are interested in such acceptors $\mathfrak{N}$ that $\mathscr{C}(\mathfrak{N}) = \mathscr{A}(\mathfrak{N}) \cup \mathscr{D}(\mathscr{A}(\mathfrak{t}(\mathfrak{N})))$ holds; now we give some properties of general acceptors.

1.10. Definition. Let $\mathfrak{N} = \langle S, V, f, I, F \rangle$ be an acceptor. We put $\mathfrak{c}(\mathfrak{N}) = \langle S \times V, S \times V, g, I \times V, F \times V \rangle$ where

$$g((s, a), (t, b)) = \begin{cases} \varnothing & \text{if } (s, a) \in S \times V, (t, b) \in S \times V, (s, a) \neq (t, b), \\ f(s, a) \times V & \text{if } (s, a) \in S \times V, (t, b) \in S \times V, \\ (s, a) = (t, b). \end{cases}$$

1.11. Lemma. *If $\mathfrak{N}$ is an arbitrary acceptor then* $\mathfrak{c}(\mathfrak{t}(\mathfrak{N})) = \mathfrak{t}(\mathfrak{c}(\mathfrak{N}))$.

Proof. Let us have $\mathfrak{N} = \langle S, V, f, I, F \rangle$. Then $\mathfrak{c}(\mathfrak{N}) = \langle S \times V, S \times V, g, I \times V, F \times V \rangle$ where g is defined according to 1.10. It follows $\mathfrak{t}(\mathfrak{c}(\mathfrak{N})) = \langle S \times V, S \times V, g, I \times V, S \times V \rangle$. Further $\mathfrak{t}(\mathfrak{N}) = \langle S, V, f, I, S \rangle$. Thus, $\mathfrak{c}(\mathfrak{t}(\mathfrak{N})) = \langle S \times V, S \times V, h, I \times V, S \times V \rangle$ and $h = g$ because h, g are defined on the basis of S, V, f in the same way.

1.12. Theorem. *Let $\mathfrak{N} = \langle S, V, f, I, F \rangle$ be an acceptor, φ a mapping of $S \times V$ into V such that $\varphi(s, a) = a$ for each $(s, a) \in S \times V$. Then* $\varphi_*(\mathscr{A}(\mathfrak{c}(\mathfrak{N}))) = \mathscr{A}(\mathfrak{N})$.

Proof. 0. We have $\mathfrak{c}(\mathfrak{N}) = \langle S \times V, S \times V, g, I \times V, F \times V \rangle$ where g is defined by 1.10. Let us have $x \in V^*$. Then there exist $m \in N$, $x_1, x_2, \ldots, x_m \in V$ such that $x = (x_i)_{1 \leqslant i < m+1}$.

1. Suppose $x \in \varphi_*(\mathscr{A}(\mathfrak{c}(\mathfrak{N})))$. If $x = \Lambda$ then $(I \times V) \cap (F \times V) \neq \varnothing$ which implies $I \cap F \neq \varnothing$; it follows $x = \Lambda \in \mathscr{A}(\mathfrak{N})$. If $x \neq \Lambda$ then $m \geqslant 1$ and there are $s_1, s_2, \ldots, s_m \in S$ such that $(s_1, x_1)(s_2, x_2) \cdots (s_m, x_m) \in \mathscr{A}(\mathfrak{c}(\mathfrak{N}))$. It implies the existence of $(t_0, y_0), (t_1, y_1), \ldots, (t_m, y_m) \in S \times V$ such that $(t_0, y_0) \in I \times V$, $(t_m, y_m) \in F \times V$ and $(t_i, y_i) \in g((t_{i-1}, y_{i-1}), (s_i, x_i))$ for $i = 1, 2, \ldots, m$. It follows $t_{i-1} = s_i$, $y_{i-1} = x_i$ for $i = 1, 2, \ldots, m$. Especially, $(t_0, y_0) \in I \times V$, $(t_m, y_m) \in F \times V$ imply $t_0 \in I$, $t_m \in F$; similarly, $(t_i, y_i) \in$

$g((t_{i-1}, y_{i-1}), (s_i, x_i)) = g((t_{i-1}, y_{i-1}), (t_{i-1}, y_{i-1})) = f(t_{i-1}, y_{i-1}) \times V$ implies $t_i \in f(t_{i-1}, y_{i-1}) = f(t_{i-1}, x_i)$ for $i = 1, 2, \ldots, m$. Thus, $x = x_1 x_2 \cdots x_m \in \mathscr{A}(\mathfrak{N})$.

We have proved $\varphi_*(\mathscr{A}(\mathfrak{c}(\mathfrak{N}))) \subseteq \mathscr{A}(\mathfrak{N})$.

2. Suppose $x \in \mathscr{A}(\mathfrak{N})$. If $x = \Lambda$ then $I \cap F \neq \varnothing$ which implies $(I \times V) \cap (F \times V) \neq \varnothing$; it follows $x = \Lambda \in \varphi_*(\mathscr{A}(\mathfrak{c}(\mathfrak{N})))$. If $x \neq \Lambda$ then $m \geqslant 1$ and there exist $s_0, s_1, \ldots, s_m \in S$ such that $s_0 \in I$, $s_m \in F$ and $s_i \in f(s_{i-1}, x_i)$ for $i = 1, 2, \ldots, m$. We put $y_i = x_{i+1}$ for $i = 0, 1, \ldots, m-1$, $y_m = x_m$. Then $(s_0, y_0) = (s_0, x_1) \in I \times V$, $(s_m, y_m) = (s_m, x_m) \in F \times V$. Similarly, $(s_i, y_i) = (s_i, x_{i+1}) \in f(s_{i-1}, x_i) \times V = g((s_{i-1}, x_i), (s_{i-1}, x_i)) = g((s_{i-1}, y_{i-1}), (s_{i-1}, x_i))$ for $i = 1, 2, \ldots, m-1$, $(s_m, y_m) = (s_m, x_m) \in f(s_{m-1}, x_m) \times V = g((s_{m-1}, x_m), (s_{m-1}, x_m)) = g((s_{m-1}, y_{m-1}), (s_{m-1}, x_m))$.

It follows that

$$(s_0, x_1)(s_1, x_2) \cdots (s_{m-1}, x_m) \in \mathscr{A}(\mathfrak{c}(\mathfrak{N})) \qquad \text{and} \qquad x_1 x_2 \cdots x_m \in \varphi_*(\mathscr{A}(\mathfrak{c}(\mathfrak{N}))).$$

We have proved $\mathscr{A}(\mathfrak{N}) \subseteq \varphi_*(\mathscr{A}(\mathfrak{c}(\mathfrak{N})))$.

1.13. Definition. Let $\mathfrak{N} = \langle S, V, f, I, F\rangle$ be an acceptor, U a set, φ a mapping of V into U. For each $(s, a) \in S \times U$, we put $g(s, a) = \bigcup_{\varphi(b)=a} f(s, b)$ and we define $\varphi\mathfrak{N} = \langle S, U, g, I, F\rangle$.

1.14. Lemma. *Let $\mathfrak{N} = \langle S, V, f, I, F\rangle$ be an acceptor, U a set, φ a mapping of V into U. Then $\mathfrak{t}(\varphi\mathfrak{N}) = \varphi\mathfrak{t}(\mathfrak{N})$.*

Proof. We have $\varphi\mathfrak{N} = \langle S, U, g, I, F\rangle$ where g is defined according to 1.13. It follows $\mathfrak{t}(\varphi\mathfrak{N}) = \langle S, U, g, I, S\rangle$. Further, $\mathfrak{t}(\mathfrak{N}) = \langle S, V, f, I, S\rangle$ and $\varphi\mathfrak{t}(\mathfrak{N}) = \langle S, U, h, I, S\rangle$ where h is defined by 1.13. But we have $h = g$ because g, h are defined on the basis of S, V, U, f, φ in the same way. Thus, $\varphi\mathfrak{t}(\mathfrak{N}) = \mathfrak{t}(\varphi\mathfrak{N})$.

1.15. Theorem. *Let $\mathfrak{N} = \langle S, V, f, I, F\rangle$ be an acceptor, U a set, φ a mapping of V into U. Then $\varphi_*(\mathscr{A}(\mathfrak{N})) = \mathscr{A}(\varphi\mathfrak{N})$.*

Proof. We have $\varphi\mathfrak{N} = \langle S, U, g, I, F\rangle$ where g is defined by 1.13. Let us have $x \in U^*$; then there are $m \in N$, $x_1, x_2, \ldots, x_m \in U$ such that $x = (x_i)_{1 \leqslant i < m+1}$.

1. Suppose $x \in \varphi_*(\mathscr{A}(\mathfrak{N}))$. If $x = \Lambda$ then $I \cap F \neq \varnothing$ which implies $x = \Lambda \in \mathscr{A}(\varphi\mathfrak{N})$. If $x \neq \Lambda$ then $m \geqslant 1$ and there exist $y_1, y_2, \ldots, y_m \in V$ such that $x_i = \varphi(y_i)$ for $i = 1, 2, \ldots, m$ and $y_1 y_2 \cdots y_m \in \mathscr{A}(\mathfrak{N})$. It implies the existence of $s_0, s_1, \ldots, s_m \in S$ such that $s_0 \in I$, $s_m \in F$ and $s_i \in f(s_{i-1}, y_i)$ for $i = 1, 2, \ldots, m$. Clearly, $f(s_{i-1}, y_i) \subseteq \bigcup_{\varphi(y)=x_i} f(s_{i-1}, y) = g(s_{i-1}, x_i)$ for $i = 1, 2, \ldots, m$. Thus, $s_i \in g(s_{i-1}, x_i)$ for $i = 1, 2, \ldots, m$ and we have $x = x_1 x_2 \cdots x_m \in \mathscr{A}(\varphi\mathfrak{N})$.

We have proved $\varphi_*(\mathscr{A}(\mathfrak{N})) \subseteq \mathscr{A}(\varphi\mathfrak{N})$.

2. Suppose $x \in \mathscr{A}(\varphi\mathfrak{N})$. If $x = \Lambda$ then $I \cap F \neq \varnothing$ which implies $\Lambda \in \mathscr{A}(\mathfrak{N})$ and $\Lambda \in \varphi_*(\mathscr{A}(\mathfrak{N}))$. If $x \neq \Lambda$ then $m \geqslant 1$ and there are $s_0, s_1, \ldots, s_m \in S$ such that $s_0 \in I$, $s_m \in F$ and $s_i \in g(s_{i-1}, x_i) = \bigcup_{\varphi(y)=x_i} f(s_{i-1}, y)$ for $i = 1, 2, \ldots, m$. Thus, for each $i = 1, 2, \ldots, m$, there exists $y_i \in V$ such that $\varphi(y_i) = x_i$ and $s_i \in f(s_{i-1}, y_i)$. It implies $y_1 y_2 \cdots y_m \in \mathscr{A}(\mathfrak{N})$ and $x = x_1 x_2 \cdots x_m = \varphi(y_1)\,\varphi(y_2) \cdots \varphi(y_m) = \varphi_*(y_1 y_2 \cdots y_m) \in \varphi_*(\mathscr{A}(\mathfrak{N}))$.

We have proved $\mathscr{A}(\varphi\mathfrak{N}) \subseteq \varphi_*(\mathscr{A}(\mathfrak{N}))$.

1.16. Definition. An acceptor $\mathfrak{N} = \langle S, V, f, I, F\rangle$ is said to have the *property* (α) if $S = V$ and $f(a, b) = \varnothing$ for each $a, b \in V$, $a \neq b$. We denote by α the class of all acceptors with the property (α).

1.17. Definition. An acceptor $\mathfrak{N} = \langle S, V, f, I, F\rangle$ is said to have the *property* (β) if it has the property (α) and if it satisfies the following conditions.

(β_1) $f(a, a) \neq \varnothing$ iff $a \in I$,

(β_2) $F = V - I$.

We denote by β the class of all acceptors with the property (β).

1.18. Definition. An acceptor $\mathfrak{N} = \langle S, V, f, I, F\rangle$ is said to have the *property* (γ) if the sets S, V are finite. We denote by γ the class of all acceptors with the property (γ).

1.19. Definition. Let ξ be a class of acceptors, V a set, $L \subseteq V^\infty$. We say that the set L is *ξ-acceptable* if there exists $\mathfrak{N} \in \xi$ such that $L = \mathscr{A}(\mathfrak{N})$. We say that L is *ξ-constructive* if there is $\mathfrak{N} \in \xi$ such that $L = \mathscr{C}(\mathfrak{N})$.

1.20. Corollary. $\mathfrak{c}(\mathfrak{N}) \in \alpha$ *for each acceptor* $\mathfrak{N}$.

1.21. Examples. We consider an acceptor $\mathfrak{N} = \langle S, V, f, I, F\rangle$.

(i) If $S = V = f = I = F = \varnothing$ then $\mathfrak{N} \in \beta \cap \gamma$ and $\mathscr{A}(\mathfrak{N}) = \varnothing = \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$.

(ii) If $S = V = I = \{a\}$ and $f = F = \varnothing$ then $\mathfrak{N} \in \alpha \cap \gamma$ and $\mathscr{A}(\mathfrak{N}) = \varnothing, \mathscr{A}(\mathfrak{t}(\mathfrak{N})) = \{\Lambda\}$.

(iii) If $S = V = I = F = \{a\}$ and $f = \varnothing$ then $\mathfrak{N} \in \alpha \cap \gamma$ and $\mathscr{A}(\mathfrak{N}) = \{\Lambda\} = \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$.

(iv) If $S = V = I = \{a\}$, $f(a, a) = \{a\}$ and $F = \varnothing$ then $\mathfrak{N} \in \beta \cap \gamma$ and $\mathscr{A}(\mathfrak{N}) = \varnothing, \mathscr{A}(\mathfrak{t}(\mathfrak{N})) = V^*$.

(v) If $S = V = I = F = \{a\}$ and $f(a, a) = \{a\}$ then $\mathfrak{N} \in \alpha \cap \gamma$ and $\mathscr{A}(\mathfrak{N}) = V^* = \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$.

2. Acceptors with the Property (α)

2.1. Theorem. *If $\mathfrak{N} \in \alpha$ then $\mathscr{C}(\mathfrak{N}) = \mathscr{A}(\mathfrak{N}) \cup \mathscr{D}(\mathscr{A}(\mathfrak{t}(\mathfrak{N})))$.*

Proof. Let us have $\mathfrak{N} \in \alpha$, $\mathfrak{N} = \langle V, V, f, I, F \rangle$, $x \in \mathscr{A}(\mathfrak{N}) \cup \mathscr{D}(\mathscr{A}(\mathfrak{t}(\mathfrak{N})))$. If $x \in \mathscr{A}(\mathfrak{N})$ then $x \in \mathscr{C}(\mathfrak{N})$. If $x \in \mathscr{D}(\mathscr{A}(\mathfrak{t}(\mathfrak{N})))$ then $x = (x_i)_{1 \leqslant i < \omega}$ for some $x_i \in V$, $1 \leqslant i < \omega$; further, there exists a sequence $(x^j)_{0 \leqslant j < \omega}$ of elements of $\mathscr{A}(\mathfrak{t}(\mathfrak{N}))$ such that $x = \lim_{j \to \omega} x^j$. Thus, there is an increasing sequence $(n^j)_{0 \leqslant j < \omega}$ of natural numbers $n^j \geqslant 1$ such that $x^j = (x_i)_{1 \leqslant i < n^j}$ for each $j \in N$. Thus, for each $j \in N$, there exists a sequence $(s_i{}^j)_{0 \leqslant i < n^j}$ of elements of V such that $s_0{}^j \in I$, $s_i{}^j \in f(s_{i-1}^j, x_i)$ for $i = 1, 2, \ldots, n^j - 1$. As ($\alpha$) holds we have $s_{i-1}^j = x_i$ for each $j \in N$ and each i, $1 \leqslant i < n^j$.

We put $s_i = x_{i+1}$ for each $i \in N$. For each $i \in N$, we choose a natural number $j(i)$ such that $n^{j(i)} > i + 1$. It follows $s_i^{j(i)} = x_{i+1} = s_i$. Especially, $s_0 = s_0^{j(0)} \in I$, $s_i = s_i^{j(i)} \in f(s_{i-1}^{j(i)}, x_i) = f(s_{i-1}, x_i)$ for each $i \in N$, $i \geqslant 1$. It follows $x = (x_i)_{1 \leqslant i < \omega} \in \mathscr{B}(\mathfrak{N}) \subseteq \mathscr{C}(\mathfrak{N})$.

We have proved $\mathscr{A}(\mathfrak{N}) \cup \mathscr{D}(\mathscr{A}(\mathfrak{t}(\mathfrak{N}))) \subseteq \mathscr{C}(\mathfrak{N})$. The assertion of our Theorem follows by 1.8.

2.2. Corollary. *If $\mathfrak{N} \in \beta$ then $C(\mathfrak{N}) = \mathscr{A}(\mathfrak{N}) \cup \mathscr{D}(\mathscr{A}(\mathfrak{t}(\mathfrak{N})))$.*

Now, we shall completely characterize sets which are accepted by acceptors with the property (α) and (β). To this aim, we need some definitions.

2.3. Definitions. If V is a set, L a subset of V^* and a an element of V then a is said to be *necessary* for L if there exist $u, v \in V^*$ such that $uav \in L$. We denote by $N(L)$ the set of all elements in V which are necessary for L.

An element $a \in V$ is said to be *final* for L if there is $u \in V^*$ such that $ua \in L$. We denote by $T(L)$ the set of all elements in V which are final for L.

2.4. *Remark.* If V, U are sets and $L \subseteq V^*$, $L \subseteq U^*$ then the set of all elements in V which are necessary (final) for L is equal to the set of all elements in U which are necessary (final) for L. Thus, these sets depend on L only; our notation $N(L)$, $T(L)$ is justified.

2.5. DEFINITION. If V is a set and L a subset of V^* then L is said to be *dense* in V^* if $N(L) = V$.

2.6. DEFINITION. If V is a set and L a subset of V^* then L is said to have the *exchange property* if the following condition is satisfied.

(e) If $x, y, u, v \in V^*$, $a \in V$, $xay \in L$ and $uav \in L$ then $xav \in L$.

2.7. DEFINITION. Let V be a set and L a subset of V^*. Then L is said to be *left hereditary* if it has the following property.

(l) If $x, y \in V^*$, $xy \in L$ then $x \in L$.

L is said to be *right hereditary* if it has the following property:

(r) If $x, y \in V^*$, $xy \in L$ then $y \in L$.

L is said to be *hereditary* if it is left hereditary and right hereditary.

2.8. DEFINITION. If V is a set and L a subset of V^* then L is said to be *left homogeneous* if it is left hereditary and has the exchange property. L is said to be *homogeneous* if it is hereditary, has the exchange property and is different from $\{\Lambda\}$.

2.9. *Remark.* It is easy to see that the property of a set $L \subseteq V^*$ to be left hereditary, right hereditary, hereditary or to have the exchange property does not depend on the set V; it is the property of L only. The same holds for the property of being left homogeneous or homogeneous.

2.10. DEFINITION. Let V be a set, $L_1 \subseteq L_2$ subsets of V^*. Then L_1 is said to be a *terminal* subset of L_2 if the following condition is satisfied.

(t) An arbitrary $x \in L_2$, $x \neq \Lambda$, is in L_1 iff there are $y \in V^*$ and $a \in T(L_1)$ such that $x = ya$.

2.11. DEFINITION. Let V be a set, $L_1 \subseteq L_2$ subsets of V^*. Then L_1 is said to be a *hyperterminal* subset of L_2 if it is a terminal subset of L_2 and if the following conditions are satisfied.

(k_1) $\Lambda \notin L_1$.

(k_2) If $a \in N(L_2)$ is such an element that $u, v \in V^*$, $uav \in L_2$ imply $v = \Lambda$ then $a \in T(L_1)$.

2.12. *Remark.* By 2.4 and 2.11, the property of a set L_1 to be a terminal or hyperterminal subset of L_2 depends on L_1 and L_2 only.

2.13. LEMMA. (i) *If* $\mathfrak{N} \in \alpha$ *then* $\mathscr{A}(\mathrm{t}(\mathfrak{N}))$ *is a left homogeneous set and* $\mathscr{A}(\mathfrak{N})$ *is its terminal subset.*

(ii) *If* $\mathfrak{N} \in \beta$ *then* $\mathscr{A}(\mathrm{t}(\mathfrak{N}))$ *is a homogeneous set and* $\mathscr{A}(\mathfrak{N})$ *is its hyperterminal subset.*

Proof. We put $\mathfrak{N} = \langle V, V, f, I, F \rangle$ where $\mathfrak{N} \in \alpha$.

1. Suppose $x, y, u, v \in V^*$, $a \in V$, $xay \in \mathscr{A}(\mathrm{t}(\mathfrak{N}))$, $uav \in \mathscr{A}(\mathrm{t}(\mathfrak{N}))$. Then there are natural numbers $m \geqslant 2$, $n \geqslant 2$, $p \geqslant 0$, $q \geqslant 0$ and some $x_i \in V$ for $1 \leqslant i < m + p$, $u_i \in V$ for $1 \leqslant i < n + q$ such that $xa = (x_i)_{1 \leqslant i < m}$, $y = (x_i)_{m \leqslant i < m+p}$, $ua = (u_i)_{1 \leqslant i < n}$, $v = (u_i)_{n \leqslant i < n+q}$. Thus there exist sequences $(s_i)_{0 \leqslant i < m+p}$, $(t_i)_{0 \leqslant i < n+q}$ of elements of V such that $s_0 \in I$, $t_0 \in I$, $s_i \in f(s_{i-1}, x_i)$ for $1 \leqslant i < m + p$, $t_i \in f(t_{i-1}, u_i)$ for $1 \leqslant i < n + q$. As $\mathfrak{N} \in \alpha$ we have $x_i = s_{i-1}$ for $1 \leqslant i < m + p$, $u_i = t_{i-1}$ for $1 \leqslant i < n + q$. Clearly, $s_{m-2} = x_{m-1} = a = u_{n-1} = t_{n-2}$. We put

$$r_i = \begin{cases} s_i & \text{for } 0 \leqslant i < m - 1, \\ t_{i+n-m} & \text{for } m - 1 \leqslant i < m + q, \end{cases}$$

$$z_i = \begin{cases} x_i & \text{for } 1 \leqslant i < m - 1, \\ u_{i+n-m} & \text{for } m - 1 \leqslant i < m + q. \end{cases}$$

We have $r_0 = s_0 \in I$, $r_i = s_i \in f(s_{i-1}, x_i) = f(r_{i-1}, z_i)$ for $1 \leqslant i < m - 1$, $r_{m-1} = t_{n-1} \in f(t_{n-2}, u_{n-1}) = f(s_{m-2}, u_{n-1}) = f(r_{m-2}, z_{m-1})$, $r_i = t_{i+n-m} \in f(t_{i+n-m-1}, u_{i+n-m}) = f(r_{i-1}, z_i)$ for $m \leqslant i < m + q$. It follows $xav = (z_i)_{1 \leqslant i < m+q} \in \mathscr{A}(\mathrm{t}(\mathfrak{N}))$. Thus, $\mathscr{A}(\mathrm{t}(\mathfrak{N}))$ has the property (e) of 2.6.

2. Suppose $x, y \in V^*$, $xy \in \mathscr{A}(\mathrm{t}(\mathfrak{N}))$. Then there is a natural number $m \geqslant 1$ and some $x_i \in V$ for $1 \leqslant i < m$ such that $xy = (x_i)_{1 \leqslant i < m}$. There exists a sequence $(s_i)_{0 \leqslant i < m}$ of elements of V such that $s_0 \in I$ and $s_i \in f(s_{i-1}, x_i)$ for $1 \leqslant i < m$. Clearly, there is a natural number n, $1 \leqslant n \leqslant m$, such that $x = (x_i)_{1 \leqslant i < n}$. The properties of the sequence $(s_i)_{0 \leqslant i < n}$ imply $x \in \mathscr{A}(\mathrm{t}(\mathfrak{N}))$. Thus, $\mathscr{A}(\mathrm{t}(\mathfrak{N}))$ has the property (1) of 2.7 and we have proved that $\mathscr{A}(\mathrm{t}(\mathfrak{N}))$ is left homogeneous.

3. Suppose $x \in \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$, $x \neq \Lambda$. Then there exist a natural number $m \geqslant 2$ and some $x_i \in V$ for $1 \leqslant i < m$ such that $x = (x_i)_{1 \leqslant i < m}$. It follows the existence of a sequence $(s_i)_{0 \leqslant i < m}$ of elements of V such that $s_0 \in I$, $s_i \in f(s_{i-1}, x_i)$ for $1 \leqslant i < m$. It implies $x_i = s_{i-1}$ for $1 \leqslant i < m$.

(A) If $x_{m-1} \in T(\mathscr{A}(\mathfrak{N}))$ then there is $k \in f(x_{m-1}, x_{m-1}) \cap F$; we put $t_i = s_i$ for $0 \leqslant i < m - 1$, $t_{m-1} = k$. Then $t_0 = s_0 \in I$, $t_i = s_i \in f(s_{i-1}, x_i) = f(t_{i-1}, x_i)$ for $1 \leqslant i < m - 1$, $t_{m-1} = k \in f(x_{m-1}, x_{m-1}) = f(s_{m-2}, x_{m-1}) = f(t_{m-2}, x_{m-1})$, $t_{m-1} \in F$. It follows $x = (x_i)_{1 \leqslant i < m} \in \mathscr{A}(\mathfrak{N})$.

(B) If $x \in \mathscr{A}(\mathfrak{N})$ then, clearly, $x_{m-1} \in T(\mathscr{A}(\mathfrak{N}))$.

We have proved that an arbitrary $x \in \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$, $x \neq \Lambda$, is in $\mathscr{A}(\mathfrak{N})$ iff there are $a \in T(\mathscr{A}(\mathfrak{N}))$ and $y \in V^*$ such that $x = ya$. Hence, $\mathscr{A}(\mathfrak{N})$ is a terminal subset of $\mathscr{A}(\mathfrak{t}(\mathfrak{N}))$.

We have proved (i).

4. Suppose $\mathfrak{N} \in \beta$.

Let us have $xy \in \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$. Then there exist a natural number $m \geqslant 1$ and some $x_i \in V$ for $1 \leqslant i < m$ such that $xy = (x_i)_{1 \leqslant i < m}$. If $y = \Lambda$ then (i) implies $y \in \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$. If $y \neq \Lambda$ then there exists $n \in N$, $1 \leqslant n < m$ such that $x = (x_i)_{1 \leqslant i < n}$, $y = (x_i)_{n \leqslant i < m}$. There is a sequence $(s_i)_{0 \leqslant i < m}$ of elements of V such that $s_0 \in I$ and $s_i \in f(s_{i-1}, x_i)$ for $1 \leqslant i < m$. If $n = 1$ then $y = xy \in \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$. If $n > 1$ then $s_n \in f(s_{n-1}, x_n) = f(s_{n-1}, s_{n-1})$ which implies $s_{n-1} \in I$. Thus, the existence of the sequence $(s_i)_{n-1 \leqslant i < m}$ with $s_{n-1} \in I$ and $s_i \in f(s_{i-1}, x_i)$ for $n \leqslant i < m$ implies $y = (x_i)_{n \leqslant i < m} \in \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$.

If $\Lambda \in \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$ then there is $a \in I$. It follows $f(a, a) \neq \varnothing$ by 1.17. Thus, $a \in \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$. It implies that $\mathscr{A}(\mathfrak{t}(\mathfrak{N})) \neq \{\Lambda\}$.

We have proved that $\mathscr{A}(\mathfrak{t}(\mathfrak{N}))$ is homogeneous.

Since $I \cap F = \varnothing$ we have $\Lambda \notin \mathscr{A}(\mathfrak{N})$ and the condition (k_1) of 2.11 is satisfied for the pair $(\mathscr{A}(\mathfrak{N}), \mathscr{A}(\mathfrak{t}(\mathfrak{N})))$.

Let $a \in N(\mathscr{A}(\mathfrak{t}(\mathfrak{N})))$ be such an element that $u, v \in V^*$, $uav \in \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$ imply $v = \Lambda$. Then there exist $x, y \in V^*$ such that $xay \in \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$. Thus, $y = \Lambda$ and there are a natural number $m \geqslant 2$ and some elements $x_i \in V$, $1 \leqslant i < m$, such that $xa = x_1 x_2 \cdots x_{m-1}$. Further, there exists a sequence $(s_i)_{0 \leqslant i < m}$ of elements of V such that $s_0 \in I$ and $s_i \in f(s_{i-1}, x_i)$ for $1 \leqslant i < m$. It follows $s_{i-1} = x_i$ for $1 \leqslant i < m$ because $\mathfrak{N} \in \alpha$. Especially, $s_{m-2} = x_{m-1} = a$ which implies $s_{m-1} \in f(s_{m-2}, x_{m-1}) = f(a, a)$ and $f(a, a) \neq \varnothing$; it follows $a \in I$ by $1.17(\beta_1)$.

Let us have $b \in f(a, a)$. If $f(b, b) \neq \varnothing$ then $ab \in \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$ contrary to our hypothesis. Thus, $f(b, b) = \varnothing$ for each $b \in f(a, a)$ which implies $b \in F$ for each $b \in f(a, a)$, by $1.17(\beta_1)$ and (β_2). It follows $f(a, a) \cap F \neq \varnothing$; hence

$xa \in \mathscr{A}(\mathfrak{N})$ and $a \in T(\mathscr{A}(\mathfrak{N}))$. It implies that the condition (k_2) of 2.11 is satisfied for the pair $(\mathscr{A}(\mathfrak{N}), \mathscr{A}(\mathfrak{t}(\mathfrak{N})))$.

Thus, $\mathscr{A}(\mathfrak{N})$ is a hyperterminal subset of $\mathscr{A}(\mathfrak{t}(\mathfrak{N}))$.

Hence, (ii) holds.

2.14. Definition. We define an operator $\mathfrak{d}$ assigning an acceptor $\mathfrak{d}(L_1, L_2, V)$ to any ordered triple (L_1, L_2, V) where V is a set, $L_2 \subseteq V^*$ a left homogeneous set and L_1 a terminal subset of L_2. The acceptor $\mathfrak{d}(L_1, L_2, V)$ is defined as follows.

Without loss of generality, we can suppose that $a \notin b$ for any $a, b \in V$. We put $a' = \{a\}$ for each $a \in V$, $V' = \{a'; a \in V\}$. Then, clearly, $'$ is a bijection of V onto V'. If $a, b \in V$ and $a' = b$ then $a \in \{a\} = b$ contrary to our hypothesis. Thus, $V \cap V' = \varnothing$.

We put $+ = V \cup V'$. Then, clearly, $+ \notin V \cup V'$.

We define $U = T(L_1)$, $U' = \{a'; a \in U\}$. Further, we put

$$W = \begin{cases} V \cup U' & \text{if } \Lambda \notin L_1, \\ V \cup U' \cup \{+\} & \text{if } \Lambda \in L_1, \end{cases}$$

$$I = \begin{cases} \{a;\ a \in V \text{ and there exists } x \in V^* \text{ with } ax \in L_2\} & \text{if } \Lambda \notin L_1, \\ \{a;\ a \in V \text{ and there exists } x \in V^* \text{ with } ax \in L_2\} \cup \{+\} & \text{if } \Lambda \in L_1, \end{cases}$$

$$F = \begin{cases} U' & \text{if } \Lambda \notin L_1, \\ U' \cup \{+\} & \text{if } \Lambda \in L_1, \end{cases}$$

$$f(a, b) = \begin{cases} \varnothing \quad \text{if } a, b \in W \text{ and either } a \neq b \text{ or } a = b \in W - V, \\ \{c;\ c \in V \text{ and there exist } u, v \in V^* \text{ with } uacv \in L_2\} \\ \qquad\qquad\qquad\qquad \text{if } b = a \in V - U, \\ \{c;\ c \in V \text{ and there exist } u, v \in V^* \text{ with } uacv \in L_2\} \cup \{a'\} \\ \qquad\qquad\qquad\qquad \text{if } b = a \in U, \end{cases}$$

$$\mathfrak{N} = \langle W, W, f, I, F\rangle, \quad \mathfrak{d}(L_1, L_2, V) = \mathfrak{N}.$$

2.15. Lemma. *Let V be a set, $L_2 \subseteq V^*$ a left homogeneous set such that $\varnothing \neq L_2 \neq \{\Lambda\}$, L_1 a terminal subset of L_2. Then the following assertions hold.*

(i) $\mathfrak{d}(L_1, L_2, V) \in \alpha$, $\mathscr{A}(\mathfrak{d}(L_1, L_2, V)) = L_1$, $\mathscr{A}(\mathfrak{t}(\mathfrak{d}(L_1, L_2, V))) = L_2$.

(ii) *If V is finite then $\mathfrak{d}(L_1, L_2, V) \in \gamma$.*

(iii) *If L_2 is a homogeneous set which is dense in V^* and if L_1 is a hyperterminal subset of L_2 then $\mathfrak{d}(L_1, L_2, V) \in \beta$.*

Proof. 0. We put $\mathfrak{d}(L_1, L_2, V) = \langle W, W, f, I, F\rangle$ where W, I, F, f are defined by 2.14. Clearly, $\mathfrak{d}(L_1, L_2, V) \in \alpha$, $V \neq \varnothing$.

1. We have $\Lambda \in L_1$ iff $I \cap F = \{+\}$; it is equivalent to $\Lambda \in \mathscr{A}(\mathfrak{d}(L_1, L_2, V))$. Clearly, $\Lambda \in L_2$, $\Lambda \in \mathscr{A}(\mathfrak{t}(\mathfrak{d}(L_1, L_2, V)))$.

2. Suppose $x \in W^*$, $x \neq \Lambda$. Then there exist a natural number $m \geqslant 2$ and some elements $x_1, x_2, \ldots, x_{m-1} \in W$ such that $x = (x_i)_{1 \leqslant i < m} = x_1 x_2 \cdots x_{m-1}$.

(A) If $x \in \mathscr{A}(\mathfrak{t}(\mathfrak{d}(L_1, L_2, V)))$ there are $s_0, s_1, \ldots, s_{m-1} \in W$ such that $s_0 \in I$ and $s_i \in f(s_{i-1}, x_i)$ for $1 \leqslant i < m$. By 0 and 2.14, we have $s_{i-1} = x_i$ and $x_i \in V$ for $1 \leqslant i < m$. Thus, $x \in V^*$. Especially, $x_1 = s_0 \in I$; by 2.14, there exists $y_1 \in V^*$ such that $x_1 y_1 \in L_2$. Suppose that $1 \leqslant i < m - 1$ and that $x_1 x_2 \cdots x_i y_i \in L_2$ for some $y_i \in V^*$. Then $x_{i+1} = s_i \in f(s_{i-1}, x_i) = f(x_i, x_i)$; by 2.14, there are $u_i, y_{i+1} \in V^*$ such that $u_i x_i x_{i+1} y_{i+1} \in L_2$. Hence, the condition (e) of 2.6 implies $x_1 x_2 \cdots x_i x_{i+1} y_{i+1} \in L_2$. By induction, it follows $x_1 x_2 \cdots x_{m-1} y_{m-1} \in L_2$ for a suitable $y_{m-1} \in V^*$. Then 2.7 (1) implies $x = x_1 x_2 \cdots x_{m-1} \in L_2$.

We have proved $\mathscr{A}(\mathfrak{t}(\mathfrak{d}(L_1, L_2, V))) \subseteq L_2$.

(B) If we have, especially, $x \in \mathscr{A}(\mathfrak{d}(L_1, L_2, V))$ then the sequence $(s_i)_{0 \leqslant i < m}$ of the part (A) can be defined in such a way that $s_{m-1} \in F$. We have proved in (A) that $\Lambda \neq x = x_1 x_2 \cdots x_{m-1} \in L_2$; further, $s_{m-1} \in F \cap f(s_{m-2}, x_{m-1}) = F \cap f(x_{m-1}, x_{m-1})$. By 2.14, it follows $s_{m-1} = x'_{m-1}$ which implies $x_{m-1} \in U = T(L_1)$. Thus, by 2.10(t), it follows $x = x_1 x_2 \cdots x_{m-1} \in L_1$.

We have proved $\mathscr{A}(\mathfrak{d}(L_1, L_2, V)) \subseteq L_1$.

(C) If $x \in L_2$ then we put $s_{i-1} = x_i$ for $1 \leqslant i < m$ and

$$s_{m-1} = \begin{cases} x'_{m-1} & \text{if } x_{m-1} \in U, \\ k & \text{where } k \in f(x_{m-1}, x_{m-1}) \text{ is arbitrary} \qquad \text{if } x_{m-1} \in V - U. \end{cases}$$

The condition $x_{m-1} \in U$ is equivalent to $x \in L_1$ by 2.10(t). In this case, we have $s_0 = x_1 \in I$, $s_i = x_{i+1} \in f(x_i, x_i) = f(s_{i-1}, x_i)$ for $1 \leqslant i < m - 1$, $s_{m-1} = x'_{m-1} \in f(x_{m-1}, x_{m-1}) = f(s_{m-2}, x_{m-1})$, $s_{m-1} = x'_{m-1} \in U' \subseteq F$, by 2.14, which implies $x \in \mathscr{A}(\mathfrak{d}(L_1, L_2, V)) \subseteq \mathscr{A}(\mathfrak{t}(\mathfrak{d}(L_1, L_2, V)))$. In the case $x_{m-1} \in V - U$, we have $s_0 = x_1 \in I$, $s_i \in f(s_{i-1}, x_i)$ for $1 \leqslant i < m - 1$ similarly as above and $s_{m-1} \in f(x_{m-1}, x_{m-1}) = f(s_{m-2}, x_{m-1})$ which implies $x \in \mathscr{A}(\mathfrak{t}(\mathfrak{d}(L_1, L_2, V)))$.

We have proved $L_2 \subseteq \mathscr{A}(\mathfrak{t}(\mathfrak{d}(L_1, L_2, V)))$, $L_1 \subseteq \mathscr{A}(\mathfrak{d}(L_1, L_2, V))$. It follows $L_1 = \mathscr{A}(\mathfrak{d}(L_1, L_2, V))$, $L_2 = \mathscr{A}(\mathfrak{t}(\mathfrak{d}(L_1, L_2, V)))$. We have (i).

3. If V is finite then $U = T(L_1) \subseteq V$ is finite, too. Thus, U' and W are finite if they are defined by 2.14. Thus, $\mathfrak{d}(L_1, L_2, V) \in \gamma$ and we have (ii).

4. Suppose that L_2 is a homogeneous set which is dense in V^* and

that L_1 is a hyperterminal subset of L_2. Then the condition (k_1) of 2.11 implies, by 2.14, that $W = V \cup U'$, $F = U'$, $I = \{a;\ a \in V$ and there is $x \in V^*$ with $ax \in L_2\}$,

$$f(a, b) = \begin{cases} \varnothing & \text{if } a, b \in W \text{ and either } a \neq b \text{ or } a = b \in W - V, \\ \{c;\ c \in V \text{ and there are } u, v \in V^* \text{ with } uacv \in L_2\} & \text{if } b = a \in V - U, \\ \{c;\ c \in V \text{ and there are } u, v \in V^* \text{ with } uacv \in L_2\} \cup \{a'\}, & \text{if } b = a \in U, \end{cases}$$

$$\mathfrak{d}(L_1, L_2, V) = \langle W, W, f, I, F \rangle.$$

By (i), $\mathfrak{d}(L_1, L_2, V) \in \alpha$.

Let $a \in V$ be arbitrary. Since $N(L_2) = V$ there exist $x, y \in V^*$ such that $xay \in L_2$. We have two possibilities:

(A) If $u, v \in V^*$, $uav \in L_2$ imply $v = \Lambda$ then $a \in T(L_1)$ because L_1 is a hyperterminal subset of L_2. By 2.14, $T(L_1) = U$. Thus, $a \in U$ and $a' \in f(a, a)$. Further, $xay \in L_2$ which implies $xa \in L_2$; it follows $a \in L_2$ because L_2 is homogeneous. Hence, $a \in I$.

(B) If there are $u, v \in V^*$, $c \in V$ such that $uacv \in L_2$ then $c \in f(a, a)$ by 2.14. It follows $acv \in L_2$ because L_2 is homogeneous. Thus, $a \in I$.

We have proved that $V \subseteq I$ and that $f(a, a) \neq \varnothing$ for each $a \in V$. Since $I \subseteq V$ we have $I = V$; since $f(a, a) = \varnothing$ for each $a \in W - V$ it follows $I = V = \{a;\ a \in W, f(a, a) \neq \varnothing\}$. By 2.14, we have $F = U' = W - V = W - I$. Thus, the conditions (β_1) and (β_2) of 1.17 are satisfied for $\mathfrak{d}(L_1, L_2, V)$ which implies that $\mathfrak{d}(L_1, L_2, V) \in \beta$.

We have proved (iii).

2.16. Theorem. *If V, L_1, L_2 are sets such that $L_1 \subseteq L_2 \subseteq V^*$ then the following assertions are equivalent.*

(i) *L_2 is a left homogeneous set and L_1 is its terminal subset.*

(ii) *There exists an acceptor $\mathfrak{N} \in \alpha$ such that $L_1 = \mathscr{A}(\mathfrak{N})$, $L_2 = \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$.*

Proof. If (i) holds and either $L_2 = \varnothing$ or $L_2 = \{\Lambda\}$ then (ii) holds by 1.21(i), (ii), (iii). If $\varnothing \neq L_2 \neq \{\Lambda\}$ then (ii) holds by 2.15(i).

Clearly, (ii) implies (i) by 2.13(i).

2.17. Theorem. *If V, L_1, L_2 are sets such that $L_1 \subseteq L_2 \subseteq V^*$ then the following assertions are equivalent.*

(i) *L_2 is a homogeneous set and L_1 is its hyperterminal subset.*

(ii) *There exists an acceptor $\mathfrak{N} \in \beta$ such that $L_1 = \mathscr{A}(\mathfrak{N}), L_2 = \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$.*

Proof. If (i) holds and $L_2 = \varnothing$ then (ii) holds by 1.21(i). If $L_2 \neq \varnothing$ then $L_2 \neq \{\Lambda\}$ by 2.8 and (ii) holds by 2.15(iii).

Clearly, (ii) implies (i) by 2.13(ii).

2.18. Corollary. *If V, L_1, L_2 are sets such that V is finite and that $L_1 \subseteq L_2 \subseteq V^*$ then the following assertions are equivalent.*

(i) *L_2 is a left homogeneous set and L_1 is its terminal subset.*

(ii) *There exists an acceptor $\mathfrak{N} \in \alpha \cap \gamma$ such that $L_1 = \mathscr{A}(\mathfrak{N})$, $L_2 = \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$.*

Proof. If (i) holds and either $L_2 = \varnothing$ or $L_2 = \{\Lambda\}$ then (ii) holds by 1.21(i), (ii), (iii). If $\varnothing \neq L_2 \neq \{\Lambda\}$ then (ii) holds by 2.15(i) and (ii).

Clearly, (ii) implies (i) by 2.16.

2.19. Corollary. *If V, L_1, L_2 are sets such that V is finite and that $L_1 \subseteq L_2 \subseteq V^*$ then the following assertions are equivalent.*

(i) *L_2 is a homogeneous set and L_1 is its hyperterminal subset.*

(ii) *There exists an acceptor $\mathfrak{N} \in \beta \cap \gamma$ such that $L_1 = \mathscr{A}(\mathfrak{N})$, $L_2 = \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$.*

Proof. If (i) holds and $L_2 = \varnothing$ then (ii) holds by 1.21(i). If $L_2 \neq \varnothing$ then $L_2 \neq \{\Lambda\}$ by 2.8 and (ii) holds by 2.15(iii) and (ii).

Clearly, (ii) implies (i) by 2.17.

2.20. Main Theorem. *If V, L are sets such that $L \subseteq V^\infty$ then the following assertions are equivalent.*

(i) *L is α-constructive.*

(ii) *There exist a left homogeneous set L_2 and its terminal subset L_1 such that $L = L_1 \cup \mathscr{D}(L_2)$.*

Proof. If (i) holds then there is $\mathfrak{N} \in \alpha$ such that $L = \mathscr{C}(\mathfrak{N})$. By 2.1, $\mathscr{C}(\mathfrak{N}) = \mathscr{A}(\mathfrak{N}) \cup \mathscr{D}(\mathscr{A}(\mathfrak{t}(\mathfrak{N})))$; by 2.16, $\mathscr{A}(\mathfrak{t}(\mathfrak{N}))$ is a left homogeneous set and $\mathscr{A}(\mathfrak{N})$ its terminal subset. Thus (ii) holds.

If (ii) holds then, by 2.16, there exists an acceptor $\mathfrak{N} \in \alpha$ such that $L_1 = \mathscr{A}(\mathfrak{N})$, $L_2 = \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$. Thus, $L = L_1 \cup \mathscr{D}(L_2) = \mathscr{A}(\mathfrak{N}) \cup \mathscr{D}(\mathscr{A}(\mathfrak{t}(\mathfrak{N}))) = \mathscr{C}(\mathfrak{N})$ by 2.1. Hence, (i) holds.

2.21. Main Theorem. *If V, L are sets such that $L \subseteq V^\infty$ then the following assertions are equivalent.*

(i) *L is β-constructive.*

(ii) *There exist a homogeneous set L_2 and its hyperterminal subset L_1 such that $L = L_1 \cup \mathscr{D}(L_2)$.*

The proof is similar to the proof of 2.20, only 2.17 is used instead of 2.16. If we use 2.18 or 2.19 instead of 2.16 we prove similarly

2.22. Main Theorem. *If V, L are sets such that V is finite and $L \subseteq V^\infty$ then the following assertions are equivalent.*

(i) *L is $\alpha \cap \gamma$-constructive.*

(ii) *There exist a left homogeneous set L_2 and its terminal subset L_1 such that $L = L_1 \cup \mathscr{D}(L_2)$.*

2.23. Main Theorem. *If V, L are sets such that V is finite and that $L \subseteq V^\infty$ then the following assertions are equivalent.*

(i) *L is $\beta \cap \gamma$-constructive.*

(ii) *There exist a homogeneous set L_2 and its hyperterminal subset L_1 such that $L = L_1 \cup \mathscr{D}(L_2)$.*

2.24. *Remark.* $\beta \cap \gamma$ is the class of all G-machines in the sense of Mezník (1972), γ is the class of all finite acceptors in the sense of Rabin and Scott (1959).

It is easy to see that each machine of Pawlak (1969) and each relational machine of Kwasowiec (1970a) can be represented in the form of an acceptor of the class α.

3. Acceptors with the Property (γ)

3.1. Theorem. *If $\mathfrak{N} \in \gamma$ then $\mathscr{C}(\mathfrak{N}) = \mathscr{A}(\mathfrak{N}) \cup \mathscr{D}(\mathscr{A}(\mathfrak{t}(\mathfrak{N})))$.*

Proof. Let us have $\mathfrak{N} \in \gamma$, $\mathfrak{N} = \langle S, V, f, I, F\rangle$, $x \in \mathscr{A}(\mathfrak{N}) \cup \mathscr{D}(\mathscr{A}(\mathfrak{t}(\mathfrak{N})))$. If $x \in \mathscr{A}(\mathfrak{N})$ then $x \in \mathscr{C}(\mathfrak{N})$. If $x \in \mathscr{D}(\mathscr{A}(\mathfrak{t}(\mathfrak{N})))$ then $x = (x_i)_{1 \leqslant i < \omega}$ for some $x_i \in V$, $1 \leqslant i < \omega$; further, there exists a sequence $(x^j)_{0 \leqslant j < \omega}$ of elements of $\mathscr{A}(\mathfrak{t}(\mathfrak{N}))$ such that $x = \lim_{j \to \omega} x^j$. Thus, there is an increasing sequence $(n^j)_{0 \leqslant j < \omega}$ of natural numbers $n^j \geqslant 1$ such that $x^j = (x_i)_{1 \leqslant i < n^j}$ for each $j \in N$. Thus, to each $j \in N$, there exists a sequence $(s_i^j)_{0 \leqslant i < n^j}$ of elements of S such that $s_0^j \in I$, $s_i^j \in f(s_{i-1}^j, x_i)$ for $i = 1, 2, \ldots, n^j - 1$.

Since the set S is finite there exists $s_0 \in S$ such that $s_0{}^j = s_0$ for an infinite set $I(0)$ of indices j. Let $k > 0$ be a natural number; suppose the existence of elements $s_0, s_1, \ldots, s_{k-1}$ of S and of an infinite set $I(k-1)$ of natural numbers such that $s_i{}^j = s_i$ for each i, $0 \leqslant i \leqslant k-1$, and each $j \in I(k-1)$. Thus, there is $s_k \in S$ such that $s_k{}^j = s_k$ for an infinite set of indices $j \in I(k-1)$. We denote by $I(k)$ this infinite set of all such indices. By induction, we construct a sequence $(s_i)_{0 \leqslant i < \omega}$ of elements of S and of infinite sets of natural numbers $(I(k))_{0 \leqslant k < \omega}$ such that $s_i{}^j = s_i$ for each i, $0 \leqslant i \leqslant k$ and each $j \in I(k)$.

We have $s_0 = s_0{}^j$ for each $j \in I(0)$ which implies $s_0 \in I$. If $k > 0$ is a natural number then $I(k)$ is an infinite set; thus, there is an index $j \in I(k)$ such that $n^j > k$. It follows $s_k = s_k{}^j \in f(s_{k-1}^j, x_k) = f(s_{k-1}, x_k)$. It implies $(x_i)_{1 \leqslant i < \omega} \in \mathscr{B}(\mathfrak{N}) \subseteq \mathscr{C}(\mathfrak{N})$.

We have proved $\mathscr{A}(\mathfrak{N}) \cup \mathscr{D}(\mathscr{A}(\mathrm{t}(\mathfrak{N}))) \subseteq \mathscr{C}(\mathfrak{N})$. The assertion follows by 1.8.

3.2. Definition. A γ-acceptable set is said to be *regular*. (See Rabin and Scott (1959) and Ginsburg (1966, Chap. II, Sect. 2.1).)

3.3. Theorem. *If V, L are sets such that V is finite and $L \subseteq V^*$ then the following assertions are equivalent*:

(i) *L is regular.*

(ii) *There exist a finite set U, an $\alpha \cap \gamma$-acceptable set $M \subseteq U^*$ and a length-preserving homomorphism Φ of U^* into V^* such that $L = \Phi(M)$.*

Proof. If (i) holds then there is $\mathfrak{N} = \langle S, V, f, I, F\rangle$, $\mathfrak{N} \in \gamma$ such that $L = \mathscr{A}(\mathfrak{N})$. We have $\mathfrak{c}(\mathfrak{N}) = \langle S \times V, S \times V, g, I \times V, F \times V\rangle \in \alpha \cap \gamma$ by 1.20 where g is defined by 1.10. We put $U = S \times V$, $M = \mathscr{A}(\mathfrak{c}(\mathfrak{N}))$; further, we put $\varphi(s, a) = a$ for each $(s, a) \in S \times V$ and $\Phi = \varphi_*$. It follows that U is finite, $M \subseteq U^*$ is $\alpha \cap \gamma$-acceptable and $\Phi(M) = \varphi_*((\mathscr{A}(\mathfrak{c}\mathfrak{N}))) = \mathscr{A}(\mathfrak{N}) = L$ by 1.12. Thus, (ii) holds.

If (ii) holds then there exist $\mathfrak{N} \in \alpha \cap \gamma$, $\mathfrak{N} = \langle W, W, f, I, F\rangle$ such that $\mathscr{A}(\mathfrak{N}) = M \subseteq U^* \cap W^* = (U \cap W)^*$ and a length-preserving homomorphism Φ of U^* into V^* such that $\Phi(M) = L$. Then there is a mapping φ of U into V such that $\varphi_* = \Phi$ by 1.1. We take an arbitrary mapping ψ of W into V such that $\psi(x) = \varphi(x)$ for each $x \in W \cap U$. Then $L = \Phi(M) = \varphi_*(M) = \psi_*(M) = \psi_*(\mathscr{A}(\mathfrak{N})) = \mathscr{A}(\psi\mathfrak{N})$ by 1.15. Clearly, $\psi\mathfrak{N} \in \gamma$ because the sets W, V are finite. Thus, L is γ-acceptable and (i) holds.

3.4. Example. Let $V = \{a\}$, $L = \{a^{2n}; n = 0, 1, \ldots\}$. Then, clearly, L is a regular language. We prove that L is not $\alpha \cap \gamma$-acceptable.

Suppose, on the contrary, that L is $\alpha \cap \gamma$-acceptable. By 2.18, L is a terminal subset of a left homogeneous set M. Thus, $L \subseteq M$ and M is left hereditary. If $n \geqslant 0$ is an arbitrary odd integer then $a^n a = a^{n+1} \in L \subseteq M$ which implies $a^n \in M$ because M is left hereditary. It follows $\{a\}^* \subseteq M$. Clearly, $T(L) = \{a\}$. Since L is a terminal subset of M we have, for an arbitrary odd integer $n \geqslant 1$, $a^n = a^{n-1}a$ and $\Lambda \neq a^n \in M$ which implies $a^n \in L$. It is a contradiction.

3.5. Definition. Let V, L be sets such that V is finite and $L \subseteq V^*$. Then L is said to be *left quasihomogeneous* if there exist a finite set U, a left homogeneous set $M \subseteq U^*$ and a length-preserving homomorphism Φ of U^* into V^* such that $L = \Phi(M)$.

3.6. Definition. Let V, L_1, L_2 be sets such that V is finite, L_2 left quasihomogeneous and $L_1 \subseteq L_2 \subseteq V^*$. Then L_1 is said to be a *quasiterminal* subset of L_2 if there are a finite set U, a left homogeneous set $M_2 \subseteq U^*$, a terminal subset M_1 of M_2 and a length-preserving homomorphism Φ of U^* into V^* such that $L_1 = \Phi(M_1)$, $L_2 = \Phi(M_2)$.

3.7. Theorem. *Let V, L_1, L_2 be sets such that V is finite and $L_1 \subseteq L_2 \subseteq V^*$. Then the following assertions are equivalent.*

(i) *L_2 is a left quasihomogeneous set and L_1 its quasiterminal subset.*

(ii) *There exists an acceptor $\mathfrak{N} \in \gamma$ such that $L_1 = \mathscr{A}(\mathfrak{N})$ and $L_2 = \mathscr{A}(\mathrm{t}(\mathfrak{N}))$.*

Proof. 1. If (i) holds then there are a finite set U, a left homogeneous set $M_2 \subseteq U^*$, a terminal subset M_1 of M_2 and a length-preserving homomorphism Φ of U^* into V^* such that $L_1 = \Phi(M_1)$, $L_2 = \Phi(M_2)$. By 1.1, there exists a mapping φ of U into V such that $\Phi = \varphi_*$. By 2.18, there is $\mathfrak{M} \in \alpha \cap \gamma$, $\mathfrak{M} = \langle W, W, f, I, F \rangle$ such that $M_1 = \mathscr{A}(\mathfrak{M})$, $M_2 = \mathscr{A}(\mathrm{t}(\mathfrak{M}))$. Clearly, $M_2 \subseteq U^* \cap W^* = (U \cap W)^*$. We put

$$\psi(x) = \begin{cases} \varphi(x) & \text{for } x \in U, \\ x & \text{for } x \in W - U. \end{cases}$$

Then ψ is a mapping of W into $(W - U) \cup V$. We put $\mathfrak{N} = \psi\mathfrak{M}$; clearly, $W, (W - U) \cup V$ are finite and $\mathfrak{N} \in \gamma$. We have $L_1 = \varphi_*(M_1) = \psi_*(\mathscr{A}(\mathfrak{M})) = \mathscr{A}(\psi\mathfrak{M}) = \mathscr{A}(\mathfrak{N})$ by 1.15, $L_2 = \varphi_*(M_2) = \psi_*(\mathscr{A}(\mathrm{t}(\mathfrak{M}))) = \mathscr{A}(\psi\mathrm{t}(\mathfrak{M})) = \mathscr{A}(\mathrm{t}(\psi\mathfrak{M})) = \mathscr{A}(\mathrm{t}\mathfrak{N})$ by 1.15 and 1.14. We have proved (ii).

2. If (ii) holds then there exists $\mathfrak{N} \in \gamma$, $\mathfrak{N} = \langle S, W, f, I, F\rangle$ such that $L_1 = \mathscr{A}(\mathfrak{N})$, $L_2 = \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$. Clearly, $L_2 \subseteq W^* \cap V^* = (W \cap V)^*$. We put $\varphi(s, a) = a$ for each $(s, a) \in S \times W$. Clearly, $\mathfrak{c}(\mathfrak{N}) \in \alpha \cap \gamma$ by 1.20. We put $\mathfrak{M} = \mathfrak{c}(\mathfrak{N})$ and we have $L_1 = \mathscr{A}(\mathfrak{N}) = \varphi_*(\mathscr{A}(\mathfrak{c}(\mathfrak{N}))) = \varphi_*(\mathscr{A}(\mathfrak{M}))$ by 1.12 and $L_2 = \mathscr{A}(\mathfrak{t}(\mathfrak{N})) = \varphi_*(\mathscr{A}(\mathfrak{c}(\mathfrak{t}(\mathfrak{N})))) = \varphi_*(\mathscr{A}(\mathfrak{t}(\mathfrak{c}(\mathfrak{N})))) = \varphi_*(\mathscr{A}(\mathfrak{t}(\mathfrak{M})))$ by 1.12 and 1.11. According to 2.18, $\mathscr{A}(\mathfrak{t}(\mathfrak{M}))$ is a left homogeneous set and $\mathscr{A}(\mathfrak{M})$ its terminal subset; further, $\mathscr{A}(\mathfrak{M}) \subseteq \mathscr{A}(\mathfrak{t}(\mathfrak{M})) \subseteq (S \times W)^*$ where $S \times W$ is a finite set. By 1.1, φ_* is a length-preserving homomorphism of $(S \times W)^*$ into W^*. It follows that L_2 is a left quasihomogeneous set and L_1 its quasiterminal subset. We have proved (i).

3.8. Main Theorem. *If V, L are sets such that V is finite and $L \subseteq V^\infty$ then the following assertions are equivalent.*

(i) *L is γ-constructive.*

(ii) *There exist a left quasihomogeneous set $L_2 \subseteq V^*$ and its quasiterminal subset L_1 such that $L = L_1 \cup \mathscr{D}(L_2)$.*

Proof. If (i) holds then there is an acceptor $\mathfrak{N} \in \gamma$ such that $L = \mathscr{C}(\mathfrak{N})$. By 3.1, we have $L = \mathscr{A}(\mathfrak{N}) \cup \mathscr{D}(\mathscr{A}(\mathfrak{t}(\mathfrak{N})))$. Then $\mathscr{A}(\mathfrak{t}(\mathfrak{N}))$ is a left quasihomogeneous set and $\mathscr{A}(\mathfrak{N})$ is its quasiterminal subset by 3.7. Since $\mathscr{D}(\mathscr{A}(\mathfrak{t}(\mathfrak{N}))) \subseteq L \subseteq V^\infty$ we have $\mathscr{A}(\mathfrak{t}(\mathfrak{N})) \subseteq V^*$ by 1.2. Thus, we have (ii).

If (ii) holds then there exists $\mathfrak{N} \in \gamma$ such that $L_1 = \mathscr{A}(\mathfrak{N})$ and $L_2 = \mathscr{A}(\mathfrak{t}(\mathfrak{N}))$ by 3.7. By 3.1, we have $L = L_1 \cup \mathscr{D}(L_2) = \mathscr{A}(\mathfrak{N}) \cup \mathscr{D}(\mathscr{A}(\mathfrak{t}(\mathfrak{N}))) = \mathscr{C}(\mathfrak{N})$ and we have (i).

Received: August 27, 1973; revised: May 1, 1974

References

Ginsburg, S. (1966), "The Mathematical Theory of Context-Free Languages," McGraw–Hill, New York–London.

Kwasowiec, W. (1970), Generable sets, *Information and Control* **17**, 257–264.

Kwasowiec, W. (1970a), Relational machines, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **18**, 545–549.

Mezník, I. (1972), G-machines and generable sets, *Information and Control* **20**, 499–509.

Rabin, M. O. and Scott, D. (1959), Finite automata and their decision problems, *IBM Systems J.* **3**, 115–125.